Solving Differential Equations Using Separation of Variables

The idea: A differential Equation can be solved easily if it has one variable (x) by taking the anti-derivative:

Example 1:

Find the general solution to the differential equation:

$$\boxed{\frac{dy}{dx} = \sin(x) + x^2}$$

To see the "separation" happen, multiply each side by dx:

$$dy = \sin(x) + x^2\, dx$$

Now, take the anti derivative of each side:

$$\int dy = \int \sin(x) + x^2\, dx$$

$$y = -\cos x + \frac{x^3}{3} + C$$

tack on to one side

Now, suppose the derivative is a function of both x and y. By separating the variables on both sides of the equation and integrating with respect to each, we can solve for the variable:

Example 2:

Find the general solution to the equation:

$$(dx)\ \frac{dy}{dx} = x^2 y\ dx)$$

Start by multiplying each side by dx:

$$dy = x^2 y\, dx$$

Now, go one step further and divide each side by y:

$$\frac{1(dy)}{y} = \frac{x^2 \cancel{y}(dx)}{\cancel{y}} \qquad \frac{1}{y}\, dy = x^2\, dx$$

Notice, we have y's on the left and x's on the right.
This lets us integrate both sides with respect to dy and dx:

$$\int \frac{1}{y}\, dy = \int x^2\, dx$$

$$\ln y = \frac{x^3}{3} + C$$

$$e^{\left(\frac{x^3}{3} + C\right)} = y$$

The last step is to solve for y. In this case, we need to use properties of logarithms to solve:

It is important to realize that not all differential equations can be solved in this way. Only derivatives that can be separated fully before integrating can be solved in this way:

A separable differential equation is in the form:

$$G(y)\,dy = F(x)\,dx$$

Example 3:

Find the specific solution to differential equation given the initial condition.

$$(dx)\ \frac{dy}{dx} = \frac{x^2}{y}(dx) \quad (y = 4 \text{ when } x = 0)$$

$$(y)\ dy = \frac{x^2}{\cancel{y}}\,dx\ (\cancel{y})$$

$$y\,dy = x^2 dx$$

$$\int y^1 dy = \int x^2 dx$$

$$\frac{y^2}{2} = \frac{x^3}{3} + C$$

$$\frac{4^2}{2} = \frac{0^3}{3} + C$$

$$8 = C$$

Plug in to solve for C

$$(2)\ \frac{y^2}{2} = \left(\frac{x^3}{3} + 8\right)(2)$$

$$y^2 = \frac{2x^3}{3} + 16$$

$$y = \sqrt{\frac{2x^3}{3} + 16}$$