HW WEEK 4 (9/17-9/21) - 1.2 (#70 and 71......refer to page 99 for support)
1.3 (#3, 6, 13, 16, 26, 28, 34, 59, 60, 73, All AP Practice Problems)
1.4 (#14, 20, 22, 36, 37, 43, 44, All AP Practice Problems)
1.5 (#4, 7, 20-22)

1.2 Limits of Functions using Properties of Limits (#70 and 71......refer to page 99 for support)

70. Let $f(x) = 2x^2 + x$. Then

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{2(x+h)^2+(x+h)-(2x^2+x)}{h} \\ &= \lim_{h\to 0} \frac{2x^2+4xh+2h^2+x+h-2x^2-x}{h} \\ &= \lim_{h\to 0} \frac{4xh+2h^2+h}{h} = \lim_{h\to 0}(4x+2h+1) = 4x+2(0)+1 = \boxed{4x+1}. \end{aligned}$$

71. Let $f(x) = \frac{2}{x}$. Then

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\frac{2}{x+h}-\frac{2}{x}}{h} = \lim_{h\to 0} \frac{\frac{2}{x+h}-\frac{2}{x}}{h}\cdot\frac{x(x+h)}{x(x+h)} = \lim_{h\to 0} \frac{2x-2(x+h)}{hx(x+h)} \\ &= \lim_{h\to 0} \frac{-2h}{hx(x+h)} = -\lim_{h\to 0} \frac{2}{x(x+h)} = -\frac{2}{x(x+0)} = \boxed{-\frac{2}{x^2}}. \end{aligned}$$

1.3 Continuity (#3, 6, 13, 16, 26, 28, 34, 59, 60, 73, All AP Practice Problems)

3. The three conditions necessary for a function f to be continuous at a number c are $\boxed{f(c) \text{ is defined}}$, $\boxed{\lim_{x\to c} f(x) \text{ exists}}$, and $\boxed{\lim_{x\to c} f(x) = f(c)}$.

6. $\boxed{\text{False}}$. If a function f is discontinuous at a number c, then it might be the case that $\lim_{x\to c} f(x)$ does not exist. However, the function could be discontinuous at c because $\lim_{x\to c} f(x)$ exists but either $f(c)$ is not defined or $\lim_{x\to c} f(x) \neq f(c)$.

13. (a) The function f is $\boxed{\text{not continuous}}$ at $c = -3$.
(b) Although f is defined at $c = -3$ with $f(-3) = 1$ and $\lim_{x\to -3} f(x) = -2$, $\boxed{\lim_{x\to -3} f(x) \neq f(-3)}$.
(c) The discontinuity at $c = -3$ is $\boxed{\text{removable}}$ because $\lim_{x\to -3} f(x)$ exists.
(d) Redefine f at $c = -3$ so that $\boxed{f(-3) = -2 = \lim_{x\to -3} f(x)}$. The resulting function will then be continuous at $c = -3$.

16. (a) The function f is $\boxed{\text{not continuous}}$ at $c = 3$.
(b) Though f is defined at $c = 3$ with $f(3) = -1$, $\boxed{\lim_{x\to 3} f(x) \text{ does not exist}}$.
(c) The discontinuity at $c = 3$ is $\boxed{\text{not removable}}$ because $\lim_{x\to 3} f(x)$ does not exist.

26. The domain of the function f is the set of all real numbers, so f is defined at $c = 1$ with $f(1) = 2$. Next,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(3x-1) = 2 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x = 2,$$

so that $\lim_{x\to 1} f(x)$ exists and is equal to 2. Finally, $\lim_{x\to 1} f(x) = f(1)$. Because all three conditions of the definition of continuity at $c = 1$ are satisfied, the function f is $\boxed{\text{continuous}}$ at $c = 1$.

28. The domain of the function f is the set of all real numbers, so f is defined at $c = 1$ with $f(1) = 2$. Next,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(3x-1) = 2 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 3x = 3,$$

so that $\lim_{x\to 1} f(x)$ does not exist. Therefore, the function f is $\boxed{\text{not continuous}}$ at $c = 1$.

34. Because

$$\lim_{x\to 3} f(x) = \lim_{x\to 3}\frac{x^2+x-12}{x-3} = \lim_{x\to 3}\frac{(x-3)(x+4)}{x-3} = \lim_{x\to 3}(x+4) = 7,$$

$\boxed{f(3) \text{ should be assigned the value } 7}$. Then $\lim_{x\to 3} f(x) = f(3)$, and the resulting function will be continuous at $c = 3$.

59. The polynomial function $f(x) = x^3 - 3x$ is continuous for all real numbers, so it is continuous on the closed interval $[-2, 2]$. Because $f(-2) = (-2)^3 - 3(-2) = -2 <$ 0 and $f(2) = 2^3 - 3(2) = 2 > 0$, the Intermediate Value Theorem guarantees that $\boxed{f \text{ must have a zero on the interval } (-2,2)}$.

60. The polynomial function $f(x) = x^4 - 1$ is continuous for all real numbers, so it is continuous on the closed interval $[-2, 2]$. Because $f(-2) = (-2)^4 - 1 = 15 > 0$ and $f(2) = 2^4 - 1 = 15 > 0$, the $\boxed{\text{Intermediate Value Theorem gives no information}}$ about the presence of a zero of f on the interval $(-2, 2)$.

73. (a) The polynomial function $x^2 + 4x$ is continuous on the set of all real numbers and is non-negative on the set $\{x|x \le -4\} \cup \{x|x \ge 0\}$. The function $f(x) = \sqrt{x^2+4x} - 2$ is therefore continuous on the set $\{x|x \le -4\} \cup \{x|x \ge 0\}$, which contains the closed interval $[0, 1]$. Because $f(0) = \sqrt{0} - 2 = -2 < 0$ and $f(1) = \sqrt{5} - 2 \approx 0.236 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(0, 1)$.

(b) Using the `FindRoot` command in the computer algebra system *Mathematica* produces the zero $\boxed{x \approx 0.828}$, rounded to three decimal places.

****DON'T WORRY ABOUT PART B FOR #73.

AP® Practice Problems

1. From the information on the graph provided,

   - f is not continuous at -1 because $f(-1)$ does not exist,
   - f is not continuous at 2 because $\lim_{x\to 2} f(x) \neq f(2)$,
   - f is not continuous at 3 because $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$. Thus, $\lim_{x\to 3} f(x)$ does not exist.

   The answer is D.

2. From the information on the graph provided,

   - $\lim_{x\to -1} f(x)$ does not exist because $\lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x)$,
   - $\lim_{x\to 1} f(x)$ does not exist because $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$,
   - $\lim_{x\to 3} f(x)$ does not exist because $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$,
   - $\lim_{x\to 2} f(x)$ exists because $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 2$,
   - $f(2) = 3$.

   Since $\lim_{x\to 2} f(x) \neq f(2)$, the function is not continuous at 2. The answer is C.

3. $\lim_{x\to -5}\left(\frac{x^2-25}{x+5}\right) = \lim_{x\to -5}\left(\frac{(x+5)(x-5)}{(x+5)}\right) = \lim_{x\to -5}(x-5) = -5-5 = -10.$

   Since $f(x)$ is continuous at –5, $f(-5) = \lim_{x\to -5}\left(\frac{x^2-25}{x+5}\right) = -10$.

   The answer is A.

4. $$\begin{aligned}\lim_{x\to 10} f(x) &= \lim_{x\to 10}\frac{\sqrt{2x+5}-\sqrt{x+15}}{x-10} = \lim_{x\to 10}\frac{(\sqrt{2x+5}-\sqrt{x+15})}{x-10}\frac{(\sqrt{2x+5}+\sqrt{x+15})}{(\sqrt{2x+5}+\sqrt{x+15})}\\ &= \lim_{x\to 10}\frac{(2x+5)-(x+15)}{(x-10)(\sqrt{2x+5}+\sqrt{x+15})} = \lim_{x\to 10}\frac{(x-10)}{(x-10)(\sqrt{2x+5}+\sqrt{x+15})}\\ &= \lim_{x\to 10}\frac{1}{(\sqrt{2x+5}+\sqrt{x+15})} = \frac{1}{(\sqrt{2(10)+5}+\sqrt{10+15})} = \frac{1}{5+5} = \frac{1}{10}.\end{aligned}$$

Since $f(x)$ is continuous at 10, $f(10) = \lim_{x\to -5} f(x)$. So, $k = \frac{1}{10}$. The answer is B.

5. Knowing that $\lim_{x\to c} f(x) = L$ does not provide any information about the value of the function at $x = c$. Refer to the graph provided in exercise 1.

- At $x = -1$, $\lim_{x\to -1} f(x) = 2$, but $f(-1)$ is not defined. So, f is not continuous at $x = -1$.
- At $x = 2$, $\lim_{x\to 2} f(x) = 1$, but $f(1) \neq 1$.

The answer is D.

6. Since f is a polynomial, it is continuous at all real numbers. Construct a table of values of f for the numbers x provided in the exercise.

x	-4	-2	-1	1	3
$f(x)$	-51	1	6	4	26

Because $f(-4) = -51$ and $f(-2) = 1$ have opposite signs, the Intermediate Value Theorem states that $f(c) = 0$ for at least one number c in the interval $(-4, -2)$. Since $f(c) = 0$ for only one real number c, then the number c must lie in the interval $(-4, -2)$. The answer is A.

7. Since f is continuous at all real numbers and $f(-8) = 3$ and $f(-1) = -4$ have opposite signs, the Intermediate Value Theorem states that $f(c) = 0$ for at least one number c in the interval $(-8, -1)$. Since $f(c) = 0$ for only one real number c, then the number c must lie in the interval $(-8, -1)$. Of the four choices, $x = -5$ is in the interval $(-8, -1)$. The answer is B.

8. For the piecewise function provided,
   - $\lim_{x\to 0} f(x) = 0$ because $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} = 0$. Function f is continuous at $x = 0$ because $\lim_{x\to 0} f(x) = f(0) = 0$.
   - $\lim_{x\to 1} f(x) = 1$ because $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} = 1$. Function f is continuous at $x = 1$ because $\lim_{x\to 1} f(x) = f(1) = 1$.
   - $\lim_{x\to 2} f(x)$ does not exist because $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$. Since $\lim_{x\to 2} f(x)$ does not exist, function f is not continuous at $x = 2$ only.

   The answer is B.

9. Since f is continuous at all real numbers and $f(-2) = 7$ is more than 2 and $f(6) = -1$ is less than 2, the Intermediate Value Theorem states that $f(c) = 2$ for at least one number c in the interval $(-2, 6)$. The answer is B.

10. The number c is $\frac{1}{2}$. Since f is continuous at all real numbers and $f(-2) = 3$ is more than 1 and $f(c) = \frac{1}{2}$ is less than 1, the Intermediate Value Theorem states that $f(x) = 1$ for at least one number x in the interval $(-2, c)$. Also, since $f(c) = \frac{1}{2}$ is less than 1 and $f(2) = 2$ is more than 1, the Intermediate Value Theorem states that $f(x) = 1$ for at least one number x in the interval $(c, 2)$. The answer is A.

11. (a) No.

    (b) Noting that $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2 - 2x + 3) = 1$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (-2x + 5) = 3$, $\lim_{x\to 1} f(x)$ does not exist because $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$. Function f is not continuous at $x = 1$ because $\lim_{x\to 1} f(x)$ does not exist.

1.4 – Limits and Continuity of Trigonometric, Exponential, and Logarithmic Functions (#14, 20, 22, 36, 37, 43, 44, All AP Practice Problems)

14. $\lim_{x\to 0} \frac{\sin x}{1 + \cos x} = \frac{\sin 0}{1 + \cos 0} = \frac{0}{1 + 1} = \boxed{0}$.

20. $\lim_{x\to 1} \ln\left(\frac{x}{e^x}\right) = \ln\left(\frac{1}{e^1}\right) = \ln e^{-1} = \boxed{-1}$.

22. $\lim_{x\to 0} \frac{1 - e^x}{1 - e^{2x}} = \lim_{x\to 0} \frac{1 - e^x}{(1 - e^x)(1 + e^x)} = \lim_{x\to 0} \frac{1}{1 + e^x} = \frac{1}{1 + e^0} = \frac{1}{1 + 1} = \boxed{\frac{1}{2}}$.

36. First, f is defined at $c = 0$ with $f(0) = 0$. Next,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \cos x = \cos 0 = 1 \quad \text{and} \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^x = e^0 = 1.$$

Because the two one-sided limits as x approaches 0 are equal to 1, it follows that $\lim_{x\to 0} f(x)$ exists and is equal to 1. However, $\lim_{x\to 0} f(x) \neq f(0)$, so $\boxed{f \text{ is not continuous at } c = 0}$.

37. First, f is defined at $c = \frac{\pi}{4}$ with $f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Next,

$$\lim_{x\to \pi/4^-} f(x) = \lim_{x\to \pi/4^-} \sin x = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \lim_{x\to \pi/4^+} f(x) = \lim_{x\to \pi/4^+} \cos x = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

Because the two one-sided limits as x approaches $\pi/4$ are equal to $\frac{\sqrt{2}}{2}$, it follows that $\lim_{x\to \pi/4} f(x)$ exists and is equal to $\frac{\sqrt{2}}{2}$. Finally, $\lim_{x\to \pi/4} f(x) = f\left(\frac{\pi}{4}\right)$, so $\boxed{f \text{ is continuous at } c = \frac{\pi}{4}}$.

43. Let $g(x) = \ln x$ and $h(x) = x - 3$. The logarithmic function g is continuous on the set $\{x | x > 0\}$, and the polynomial function h is continuous on the set of all real numbers. As f is the quotient of the functions g and h and the only value x for which $h(x) = 0$ is $x = 3$, it follows that the function $\boxed{f \text{ is continuous on the set } \{x | x > 0, x \neq 3\}}$.

44. Let $g(x) = \ln x$ and $h(x) = x^2 + 1$. The logarithmic function g is continuous on the set $\{x | x > 0\}$, and the polynomial function h is continuous on the set of all real numbers. As the function f is the composition $g(h(x))$ and g is continuous at $h(x)$ for all x because $x^2 + 1 \geq 1 > 0$ for any real number x, it follows that $\boxed{f \text{ is continuous on the set of all real numbers}}$

AP® Practice Problems

1. Let $t = 2x$. As $x \to 0$, $t \to 0$. Then $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} \frac{\sin(2x)}{2x} = \lim\limits_{t\to 0} \frac{\sin(t)}{t} = 1$.

   Since g is continuous at $x = 0$, $\lim\limits_{x\to 0} g(x) = g(0) = 1$. Since $g(0) = k$, k must equal 1.

   The answer is C.

2. $\lim\limits_{x\to 0} \dfrac{\sin(4x)}{2x} = 2 \lim\limits_{x\to 0} \dfrac{\sin(4x)}{4x}$

   Let $t = 4x$. As $x \to 0$, $t \to 0$. Then $\lim\limits_{x\to 0} \frac{\sin(4x)}{2x} = 2 \lim\limits_{t\to 0} \frac{\sin(4x)}{4x} = 2 \lim\limits_{t\to 0} \frac{\sin(t)}{t} = 2(1) = 2$.

   The answer is D.

3. $\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to -2^-} \left(x^3 + 2x^2\right) = (-2)^3 + 2(-2)^2 = -8 + 8 = 0$.

   $\lim\limits_{x\to -2^+} f(x) = \lim\limits_{x\to -2^+} \left(e^{2x+4}\right) = e^{2(-2)+4} = e^0 = 1$.

   $\lim\limits_{x\to -2} f(x)$ does not exist because $\lim\limits_{x\to -2^-} f(x) \neq \lim\limits_{x\to -2^+} f(x)$.

   The answer is D.

4. Use the identity $\sin^2\theta + \cos^2\theta = 1$. Let $t = 3x$. As $x \to 0$, $t \to 0$.

$$\lim_{x\to 0}\frac{1-\cos^2(3x)}{x^2} = \lim_{x\to 0}\frac{\sin^2(3x)}{x^2} = 9\lim_{x\to 0}\frac{\sin^2(3x)}{9x^2} = 9\lim_{x\to 0}\left(\frac{\sin(3x)}{3x}\right)^2$$
$$= 9\left(\lim_{x\to 0}\frac{\sin(3x)}{3x}\right)^2 = 9\left(\lim_{t\to 0}\frac{\sin(t)}{t}\right)^2 = 9(1)^2 = 9.$$

The answer is D.

5. Function $f(x) = x^{1/3}$ and $h(x) = e^{-x}$ are continuous for all real numbers x. For all real numbers c, $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} h(x) = h(c)$. Function $g(x) = \sec x$ is not defined for $x = n\pi$ for any integer n. Therefore, $g(x) = \sec x$ is not continuous at $x = n\pi$ for any integer n. The answer is C.

6. Use the identity $\frac{1}{\csc\theta} = \sin\theta$.

$\lim_{x\to 0}\frac{1}{x\csc x} = \lim_{x\to 0}\frac{1}{x}\frac{1}{\csc x} = \lim_{x\to 0}\frac{1}{x}\sin x = \lim_{x\to 0}\frac{\sin x}{x} = 1.$ The answer is C.

7. To find $\lim_{x\to\pi/3}\frac{\sin(x-\frac{\pi}{3})}{x-\frac{\pi}{3}}$, let $t = x - \frac{\pi}{3}$. As $x \to \frac{\pi}{3}, t \to 0$.

$\lim_{x\to\pi/3}\frac{\sin(x-\frac{\pi}{3})}{x-\frac{\pi}{3}} = \lim_{t\to 0}\frac{\sin(t)}{t} = 1.$ The answer is C.

8. Use the identity $\sin^2\theta + \cos^2\theta = 1$.

$$\lim_{x\to 0}\frac{1-\cos x}{3\sin^2 x} = \frac{1}{3}\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x} = \frac{1}{3}\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{\sin^2 x(1+\cos x)} = \frac{1}{3}\lim_{x\to 0}\frac{1-\cos^2 x}{\sin^2 x(1+\cos x)}$$
$$= \frac{1}{3}\lim_{x\to 0}\frac{\sin^2 x}{\sin^2 x(1+\cos x)} = \frac{1}{3}\lim_{x\to 0}\frac{1}{1+\cos x} = \frac{1}{3}\left(\frac{1}{1+\cos 0}\right) = \frac{1}{3}\left(\frac{1}{1+1}\right) = \frac{1}{6}.$$

The answer is A.

9. To determine whether $\lim_{x\to 3} f(x)$ exists, check that $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$.

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \ln x = \ln 3.$
$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (2x-3)\ln 3 = (2(3)-3)\ln 3 = 3\ln 3.$

Since $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$, $\lim_{x\to 3} f(x)$ does not exist.

The answer is D.

10. $\lim\limits_{x\to 0}\frac{\tan(2x)}{3x}=\frac{1}{3}\lim\limits_{x\to 0}\left(\frac{1}{x}\cdot\tan(2x)\right)=\frac{1}{3}\lim\limits_{x\to 0}\left(\frac{1}{x}\cdot\frac{\sin(2x)}{\cos(2x)}\right)$

$=\frac{1}{3}\lim\limits_{x\to 0}\left(\frac{\sin(2x)}{x}\cdot\frac{1}{\cos(2x)}\right)=\frac{2}{3}\lim\limits_{x\to 0}\left(\frac{\sin(2x)}{2x}\cdot\frac{1}{\cos(2x)}\right)$

$=\frac{2}{3}\lim\limits_{x\to 0}\frac{\sin(2x)}{2x}\lim\limits_{x\to 0}\frac{1}{\cos(2x)}.$

Let $t=2x$. As $x\to 0$, $t\to 0$.

$$\lim_{x\to 0}\frac{\tan(2x)}{3x}=\frac{2}{3}\lim_{x\to 0}\frac{\sin(2x)}{2x}\lim_{x\to 0}\frac{1}{\cos(2x)}=\frac{2}{3}(1)(1)=\frac{2}{3}.$$

The answer is C.

11. Use the Squeeze Theorem to find $\lim\limits_{x\to 0}x^3\sin\frac{1}{x}$.

Begin with the inequality $\left|\sin\frac{1}{x}\right|\le 1$. Multiply both sides by $\left|x^3\right|$ to obtain $\left|x^3\right|\left|\sin\frac{1}{x}\right|\le\left|x^3\right|$.

Using the property of absolute values, $\left|x^3\sin\frac{1}{x}\right|\le\left|x^3\right|$. Thus, $-\left|x^3\right|\le x^3\sin\frac{1}{x}\le\left|x^3\right|$.

Now use the Squeeze Theorem with $f(x)=-\left|x^3\right|$, $g(x)=x^3\sin\frac{1}{x}$, and $h(x)=\left|x^3\right|$.

Since $f(x)\le g(x)\le h(x)$, $\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}\left(-\left|x^3\right|\right)=0$ and $\lim\limits_{x\to 0}g(x)=\lim\limits_{x\to 0}\left|x^3\right|=0$, it follows that $\lim\limits_{x\to 0}g(x)=\lim\limits_{x\to 0}x^3\sin\frac{1}{x}=0$. The answer is B.

1.5 - Infinite Limits; Limits at Infinity; Asymptotes (#4, 7, 20-22)

4. If $\lim\limits_{x\to 4}f(x)=\infty$, then the line $x=4$ is a $\boxed{\textbf{vertical}}$ asymptote of the graph of f.

7. (a) $\lim\limits_{x\to -\infty}e^x=\boxed{0}$.

   (b) $\lim\limits_{x\to \infty}e^x=\boxed{\infty}$.

   (c) $\lim\limits_{x\to \infty}e^{-x}=\boxed{0}$.

20. As x approaches -3 from the right, the graph of f becomes unbounded in the negative direction. Thus, $\boxed{\lim\limits_{x\to -3^+}f(x)=-\infty}$.

21. As x approaches 0 from the left, the graph of f approaches the origin. Thus, $\boxed{\lim\limits_{x\to 0^-}f(x)=0}$.

22. As x approaches 0 from the right, the graph of f becomes unbounded in the positive direction. Thus, $\boxed{\lim\limits_{x\to 0^+}f(x)=\infty}$.